# STATISTICAL INFERENCE FOR FRACTIONAL DIFFUSION PROCESSES

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# Estimation for cusp

- Suppose $X_1, X_2, \ldots, X_n$ are i.i.d. random variables with the probability density function

$$f(x,\theta) = C(\lambda) e^{-|x-\theta|^{\lambda}}, \quad 0 < \lambda < \frac{1}{2}.$$

Let $L_n(\theta) = \sum_{i=1}^{n} \log f(X_i, \theta)$. Apply the transformation $t = n^{1/(2\lambda+1)}(\theta - \theta_0)$ or equivalently $\theta = \theta_0 + tn^{-1/(2\lambda+1)}$. Consider the log-likelihood ratio process

$$M_n(\theta) = L_n(\theta) - L_n(\theta_0)$$

or equivalently $Z_n(t) = M_n(\theta_0 + tn^{-1/(2\lambda+1)})$.

# Estimation for cusp

- **Theorem :** The sequence of stochastic processes $\{Z_n(t), -\tau \leq t \leq \tau\}$ converge in distribution to the Gaussian process $\{Z(t), -\tau \leq t \leq \tau\}$ with mean

$$E[Z(t)] = -K(\lambda)|t|^{2\lambda+1}$$

and

$$Cov(Z(t), Z(s)) = K(\lambda)[|t|^{2\lambda+1} + |s|^{2\lambda+1} - |t-s|^{2\lambda+1}].$$

# Estimation for cusp

- "Asymptotic Distributions in Some Non-regular Statistical Problems" was the topic of my Ph.D. Dissertation prepared under the guidance of Prof. Herman Rubin at Michigan State University in 1966. One of the non-regular problems studied in the dissertation was the problem of estimation of the location of cusp of a continuous density. The approach adapted was to study the limiting distribution of the log-likelihood ratio process and then obtain the asymptotic properties of the maximum likelihood estimator.

# Estimation for cusp

- It turned out that the limiting process is a special type of a non-stationary gaussian process. The name fractional Brownian motion was not in vogue in those years and the limiting process is nothing but a functional shift of a fractional Brownian motion.

# Self-similar processes

- Long range dependence phenomenon is said to occur in a stationary time series $\{X_n, n \geq 0\}$ if the $Cov(X_0, X_n)$ of the time series tends to zero as $n \to \infty$ and yet the condition

$$\sum_{n=0}^{\infty} |Cov(X_0, X_n)| = \infty$$

holds. In other words the covariance between $X_0$ and $X_n$ tends to zero but so slowly that their sums diverge.

# Self-similar processes

- Long range dependence is also related to the concept of self-similarity for a stochastic process in that the increments of a self-similar process with stationary increments exhibit long range dependence.

# Self-similar processes

- A real-valued stochastic process $Z = \{Z(t), -\infty < t < \infty\}$ is said to be *self-similar* with index $H > 0$ if, for any $a > 0$,

$$\{Z(at), -\infty < t < \infty\} = \{a^H Z(t), -\infty < t < \infty\}$$

where the equality indicates the equality of the finite dimensional distributions of the process on the right side of the equation with the corresponding finite dimensional distributions of the process on the left side of the equation .

# Self-similar processes

- The index $H$ is called the *scaling exponent* or the *fractal index* or the *Hurst parameter* of the process. If $H$ is the scaling exponent of a self-similar process $Z$, then the process $Z$ is called $H$-*self similar* process or $H$-ss process for short. It can be checked that a non-degenerate $H$-ss process cannot be a stationary process.

# Self-similar processes

- If $\{Z(t), t > 0\}$ is a $H$-ss process, then the process

$$Y(t) = e^{-tH} Z(e^t), -\infty < t < \infty$$

  is a stationary process.
- Conversely if $Y = \{Y(t), -\infty < t < \infty\}$ is a stationary process, then $Z = \{t^H Y(\log t), t > 0\}$ is a $H$-ss process. Suppose $Z = \{Z(t), -\infty < t < \infty\}$ is a $H$-ss process with finite variance and stationary increments,then the following properties hold:

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# Self-similar processes

- (i) $Z(0) = 0$ a.s;
- (ii) If $H \neq 1$, then $E(Z(t)) = 0, -\infty < t < \infty$;
- (iii) $Z(-t)$ and $-Z(t)$ have the same distribution;
- (iv)$E(Z^2(t)) = |t|^{2H}E(Z^2(1))$;
- (v)The covariance function $\Gamma_H(t,s)$ of the process $Z$ is given by

$$\Gamma_H(t,s) = \frac{1}{2}\{|t|^{2H} + |s|^{2H} - |t-s|^{2H}\}E(Z^2(1)).$$

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# Self-similar processes

- (vi)The self-similarity parameter, also called the scaling exponent or fractal index $H$, is less than or equal to one.
- (vii) If $H = 1$, then $Z(t) = t\, Z(1)$ a.s. for $-\infty < t < \infty$.
- (viii) Let $0 < H \leq 1$. Then the function $R_H(s,t) = \{|t|^{2H} + |s|^{2H} - |t-s|^{2H}\}$ is nonnegative definite.

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# Self-similar processes

- It was observed that there are some phenomena which exhibit self-similar behaviour locally but the nature of self-similarity changes as the phenomenon evolves. It was suggested that the parameter $H$ must be allowed to vary as function of time for modeling such data. Such processes are called *locally self-similar*.

# fractional Brownian motion

- A Gaussian process $H$-ss process

$$W^H = \{W_H(t), -\infty < t < \infty\}$$

with stationary increments and with fractal index $0 < H < 1$ is called a *fractional Brownian motion* (fBm).
It is said to be standard if $Var(W^H(1)) = 1$.

# fractional Brownian motion

- For any $0 < H < 1$, there exists a version of the fBm for which the sample paths are continuous with probability one but are not differentiable even in the $L^2$-sense. The continuity of the sample paths follows from the Kolmogorov's continuity condition and the fact that

$$E|W^H(t_2) - W^H(t_1)|^\alpha = E|W^H(1)|^\alpha |t_2 - t_1|^{\alpha H}$$

from the property that the fBm is a $H$-ss process with stationary increments. Choose $\alpha$ such that $\alpha H > 1$ to satisfy the Kolmogorov's continuity condition.

# fractional Brownian motion

- Furthermore

$$E\left|\frac{W^H(t_2) - W^H(t_1)}{t_2 - t_1}\right|^2 = E[W^H(1)^2]|t_2 - t_1|^{2H-2}$$

and the last term tends to infinity as $t_2 \to t_1$ since $H < 1$. Hence the paths of the fBm are not $L^2$-differentiable.

# fractional Brownian motion

- Consider the fBm $W^H$ on the interval $[0, T]$ with Hurst index $H$. Then

$$
\begin{aligned}
\lim_{n\to\infty} \sum_{j=0}^{2^n-1} |W^H(\frac{j+1}{2^n}T) - W^H(\frac{j}{2^n}T)|^p &= 0 \text{ a.s if } pH > 1 \\
&= \infty \text{ a.s if } pH < 1 \\
&= T \text{ a.s if } pH = 1
\end{aligned}
$$

# fractional Brownian motion

- (cf. Decreusefond and Ustunel(1999), Potential Analysis, 10, 177-214).
  (a) If $H = \frac{1}{2}$ and $p = 2$, then we have the Baxter's theorem.
- (b) If $H > \frac{1}{2}$ and $p = 2$, then the quadratic variation of the process $W^H$ is zero a.s. and the process is called a Dirichlet process.
- (c) If $H < \frac{1}{2}$ and $p = 2$, then the process $W^H$ has infinite quadratic variation a.s.
- (d) If $pH = 1$, then the process $W^H$ has finite $p$-th variation a.s.

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# fractional Brownian motion

- Furthermore

$$\sum_{j=0}^{n-1} |W^H(t_{j+1}^{(n)}) - W^H(t_j^{(n)})|^2$$

tends to zero in probability as $n$ tends to infinity for any sequence of subdivisions $0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_n^{(n)} = T$ of the interval $[0, T]$, such that the norm of the subdivision goes to zero as $n \to \infty$. Hence the process $W^H$ is not a semimartingale cf. Liptser and Shiryayev (Theory of Martingales, 1986.)

# fractional Brownian motion

- Representation of standard fBm $\{Z(t), t \geq 0\}$ with Hurst index $H$ : For $0 \leq s \leq t$,

$$\begin{aligned} Z(t) - Z(s) &= C_H\Big(\int_s^t (t-u)^{H-\frac{1}{2}} W(du) \\ &\quad + \int_{-\infty}^t [(t-u)^{H-\frac{1}{2}} - (s-u)^{H-\frac{1}{2}}] W(du)\Big) \end{aligned}$$

where

$$C_H = \Big[\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)}\Big]^{1/2}.$$

# fractional Gaussian noise

- Suppose $\{Z(t), -\infty < t < \infty\}$ is a $H$-self similar process with stationary increments and $0 < H < 1$. Define

$$X_k = Z(k+1) - Z(k), -\infty < k < \infty.$$

If the process $Z$ is a fBm, then the process $\{X_k\}$ is called **fractional Gaussian noise.** The autocovariance function of the process $\{X_k\}$ is

$$\gamma(k) = E(X_i X_{k+i}) = \frac{\sigma^2}{2}(|k+1|^{2H} + |k-1|^{2H} - 2|k|^{2H}), \quad E[X_k^2] = \sigma^2.$$

# fractional Gaussian noise

- Suppose $k \neq 0$. Then $\gamma(k) = 0$ if $H = \frac{1}{2}, \gamma(k) < 0$ if $0 < H < \frac{1}{2}$, and $\gamma(k) > 0$ if $\frac{1}{2} < H < 1$. This follows by noting that the function $f(x) = x^{2H}$ is strictly convex if $\frac{1}{2} < H < 1$ and strictly concave if $0 < H < \frac{1}{2}$ for $x > 0$. If $\frac{1}{2} < H < 1$, then $\sum_{\infty}^{\infty} \gamma(k) = \infty$ and the process $\{X_k, -\infty < k < \infty\}$ exhibits long range dependence.

# Stochastic integral with respect to a fBm

- (i)If $Y$ is a simple process,that is

$$Y_t = \sum_{j=1}^{k} X_j I_{(t_{j-1},t_j]}(t),$$

define

$$\int_{-\infty}^{\infty} Y_t dZ_t = \sum_{j=1}^{k} X_j (Z(t_j) - Z(t_{j-1})).$$

# Stochastic integral with respect to a fBm

- (ii) If the process $Y$ is of locally bounded variation, then define
$$\int_a^b Y_t dZ_t = Y_b Z_b - Y_a Z_a - \int_a^b Z_t dY_t$$
using the integration by parts formula and interpreting the integral on the right side as the Lebesgue-Stieltjes integral.

# Stochastic integral with respect to a fBm

- (iii) For any non-random (deterministic function) $f \in L^2(R) \cap L^1(R)$, define

$$\int_{-\infty}^{\infty} f(t)dZ_t = C_H(H-\frac{1}{2})\int_{-\infty}^{\infty}[\int_{\tau}^{\infty}(t-\tau)^{H-\frac{1}{2}}f(t)dt]W(d\tau).$$

# Stochastic integral with respect to a fBm

- In general, it is not possible to define a stochastic integral of a random function with respect to a fractional Brownian motion as in the case of Ito integrals for non-anticipating processes using limits of Riemann type partial sums. This is due to the fact that fBm is not a semimartingale. There are other approaches to define a stochastic integral using the notion of Wick product but they do not seem to be useful for interpretation for stochastic modeling.

# Stochastic integral with respect to a fBm

- There is a nice covariance formula in this context.

$$E\left[\int_{-\infty}^{\infty} f(t)dZ_t \int_{-\infty}^{\infty} g(t)dZ_t\right]$$

$$= H(2H-1)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(s)g(t)|t-s|^{2H-2}dtds.$$

# Fractional diffusion processes

- Consider the stochastic process $Y = \{Y_t, t \geq 0\}$ defined by the stochastic integral equation

$$Y_t = \int_0^t C(s)ds + \int_0^t B(s)dW_s^H, t \geq 0$$

where $C = \{C(t), t \geq 0\}$ is an adapted process with respect to the underlying filtration and the function $B(.)$ is a non-vanishing non-random function.

# Fractional diffusion processes

- Write the above integral equation as a stochastic differential equation

$$dY_t = C(t)dt + B(t)dW_t^H, t \geq 0$$

driven by the fractional Brownian motion $W^H$. Such a process is called a **fractional diffusion process.** The process $Y$ is not a semimartingale. However one can associate a semimartingale $Z = \{Z_t, t \geq 0\}$ which is called a *fundamental semimartingale* such that the natural filtration of the process $Z$ coincides with the natural filtration of the process $Y$ (Kleptsyna et al. (2000)).

# Fractional diffusion processes

- Define, for $0 < s < t$,

$$k_H = 2H\ \Gamma(\frac{3}{2} - H)\ \Gamma(H + \frac{1}{2}),$$

$$\kappa_H(t,s) = k_H^{-1} s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H},$$

$$\lambda_H = \frac{2H\,\Gamma(3-2H)\ \Gamma(H+\frac{1}{2})}{\Gamma(\frac{3}{2}-H)},$$

$$w_t^H = \lambda_H^{-1} t^{2-2H},$$

and

$$M_t^H = \int_0^t \kappa_H(t,s)dW_s^H, t \geq 0.$$

# Fractional diffusion processes

- The process $M^H$ is a Gaussian martingale, called the *fundamental martingale* and its quadratic variation $< M_t^H > = w_t^H$. Further more the natural filtration of the martingale $M^H$ coincides with the natural filtration of the fBm $W^H$. In fact the stochastic integral

$$\int_0^t B(s) dW_s^H$$

can be represented in terms of the stochastic integral with respect to the martingale $M^H$.

# Fractional diffusion processes

- For a measurable function $f$ on $[0, T]$, let

$$K_H^f(t,s) = -2H \frac{d}{ds} \int_s^t f(r) r^{H-\frac{1}{2}} (r-s)^{H-\frac{1}{2}} dr, 0 \leq s \leq t$$

  when the derivative exists in the sense of absolute continuity with respect to the Lebesgue measure.

- Let $M^H$ be the fundamental martingale associated with the fBm $W^H$. Then

$$\int_0^t f(s) dW_s^H = \int_0^t K_H^f(t,s) dM_s^H, t \in [0, T]$$

  a.s $[P]$ whenever both sides are well defined.

# Fractional diffusion processes

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# Fractional diffusion processes

- Suppose the sample paths of the process $\{\frac{C(t)}{B(t)}, t \geq 0\}$ are smooth enough so that

$$Q_H(t) = \frac{d}{dw_t^H} \int_0^t \kappa_H(t,s) \frac{C(s)}{B(s)} ds, t \in [0,T]$$

is well-defined where

$$w_t^H = \lambda_H^{-1} t^{2-2H}, \quad \kappa_H(t,s) = k_H^{-1} s^{\frac{1}{2}-H} (t-s)^{\frac{1}{2}-H}.$$

Kleptsyna et al. (2000) associates a *fundamental semimartingale* $Z$ associated with the process $Y$ such that the natural filtration of the processes $Z$ coincides with the natural filtration $Y$.

# Fractional diffusion processes

- **Theorem :** Suppose the sample paths of the process $Q_H$ belong $P$-a.s to $L^2([0,T], dw^H)$. Let the process $Z = \{Z_t, t \in [0,T]\}$ be defined by

$$Z_t = \int_0^t \frac{\kappa_H(t,s)}{B(s)} dY_s.$$

Then the following results hold: (i) The process $Z$ is a semimartingale with the decomposition

$$Z_t = \int_0^t Q_H(s) dw_s^H + M_t^H.$$

# Fractional diffusion processes

- (ii) the process $Y$ admits the representation

$$Y_t = \int_0^t K_H^B(t,s)dZ_s.$$

# Fractional diffusion processes

- (iii) the natural filtrations of the processes $Z$ and $Y$ coincide. Kleptsyna et al. (2000) derived the following Girsanov type formula .

# Fractional diffusion processes

- **Theorem :** Define

$$\Lambda_H(T) = \exp\{-\int_0^T Q_H(t)dM_t^H - \frac{1}{2}\int_0^t Q_H^2(t)dw_t^H\}.$$

Suppose that $E(\Lambda_H(T)) = 1$. Then the measure $P^* = \Lambda_H(T)P$ is a probability measure and the probability measure of the process $Y$ under $P^*$ is the same as that of the process $V$ defined by

$$V_t = \int_0^t B(s)dW_s^H, 0 \leq t \leq T.$$

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# Linear SDE driven by fBm

- Statistical inference for diffusion type processes satisfying stochastic differential equations driven by Wiener processes have been studied earlier. There has been a recent interest to study similar problems for stochastic processes driven by a fractional Brownian motion, hereafter called fractional diffusion processes, for modeling stochastic phenomena with possible long range dependence.

# Linear SDE driven by fBm

- Fractional Ornstein-Uhlenbeck type process is a fractional analogue of the Ornstein-Uhlenbeck process, that is, a continuous time first order autoregressive process $X = \{X_t, t \geq 0\}$ which is the solution of the one-dimensional homogeneous linear stochastic differential equation driven by a fractional Brownian motion (fBm) $W^H = \{W_t^H, t \geq 0\}$ with Hurst parameter $H \in [1/2, 1)$. Such a process is the unique Gaussian process satisfying the linear integral equation

$$X_t = \theta \int_0^t X_s ds + \sigma W_t^H, t \geq 0.$$

- The problem is to estimate the parameters $\theta$ and $\sigma^2$ based on the observation $\{X_s, 0 \leq s \leq T\}$ and study their properties as $T \to \infty$. Let us consider a slightly more general stochastic differential equation

$$dX(t) = [a(t, X(t)) + \theta\, b(t, X(t))]dt + \sigma(t) dW_t^H, X(0) = 0, t \geq 0.$$

Let

$$C(\theta, t) = a(t, X(t)) + \theta\, b(t, X(t)), t \geq 0.$$

- Define

$$Q_{H,\theta}(t) = \frac{d}{dw_t^H} \int_0^t \kappa_H(t,s) \frac{C(\theta,s)}{\sigma(s)} ds, t \geq 0$$

and

$$Z_t = \int_0^t \frac{\kappa_H(t,s)}{\sigma(s)} dX_s, t \geq 0.$$

# Linear SDE driven by fBm

- Then the process $Z = \{Z_t, t \geq 0\}$ is semimartingale with the decomposition

$$Z_t = \int_0^t Q_{H,\theta}(s) dw_s^H + M_t^H$$

where $M^H$ is the fundamental martingale and the process $X$ admits the representation

$$X_t = \int_0^t K_H^{\sigma}(t,s) dZ_s$$

where the function $K_H^{\sigma}(.,.)$ can be explicitly specified.

# Linear SDE driven by fBm

- Let $P_\theta^T$ be the measure induced by the process $\{X_t, 0 \leq t \leq T\}$ when $\theta$ is the true parameter. The Radon-Nikodym derivative of $P_\theta^T$ with respect to $P_0^T$ is given by

$$\frac{dP_\theta^T}{dP_0^T} = \exp\left[\int_0^T Q_{H,\theta}(s) dZ_s - \frac{1}{2}\int_0^T Q_{H,\theta}^2(s) dw_s^H\right].$$

# Linear SDE driven by fBm

- **Maximum likelihood estimation :** Consider the problem of estimation of the parameter $\theta$ based on the observation of the process $X = \{X_t, 0 \leq t \leq T\}$ and study its asymptotic properties as $T \to \infty$.
  **Strong consistency :** Let $L_T(\theta)$ denote the Radon-Nikodym derivative $\frac{dP_\theta^T}{dP_0^T}$. The maximum likelihood estimator (MLE) $\hat{\theta}_T$ is defined by the relation

$$L_T(\hat{\theta}_T) = \sup_{\theta \in \Theta} L_T(\theta).$$

# Linear SDE driven by fBm

- Note that

$$
\begin{aligned}
Q_{H,\theta}(t) &= \frac{d}{dw_t^H}\int_0^t \kappa_H(t,s)\frac{C(\theta,s)}{\sigma(s)}ds \\
&= \frac{d}{dw_t^H}\int_0^t \kappa_H(t,s)\frac{a(s,X(s))}{\sigma(s)}ds \\
&\quad +\theta\frac{d}{dw_t^H}\int_0^t \kappa_H(t,s)\frac{b(s,X(s))}{\sigma(s)}ds \\
&= J_1(t)+\theta J_2(t).(say)
\end{aligned}
$$

# Linear SDE driven by fBm

- Then
$$\log L_T(\theta) = \int_0^T (J_1(t)+\theta J_2(t))dZ_t - \frac{1}{2}\int_0^T (J_1(t)+\theta J_2(t))^2 dw_t^H$$
and the likelihood equation is given by
$$\int_0^T J_2(t)dZ_t - \int_0^T (J_1(t)+\theta J_2(t))J_2(t)dw_t^H = 0.$$
Hence the MLE $\hat{\theta}_T$ of $\theta$ is given by
$$\hat{\theta}_T = \frac{\int_0^T J_2(t)dZ_t + \int_0^T J_1(t)J_2(t)dw_t^H}{\int_0^T J_2^2(t)dw_t^H}.$$

# Linear SDE driven by fBm

- Let $\theta_0$ be the true parameter. Using the fact that

$$dZ_t = (J_1(t) + \theta_0 J_2(t))dw_t^H + dM_t^H,$$

it can be shown that

$$\frac{dP_\theta^T}{dP_{\theta_0}^T} = \exp[(\theta-\theta_0)\int_0^T J_2(t)dM_t^H - \frac{1}{2}(\theta-\theta_0)^2\int_0^T J_2^2(t)dw_t^H].$$

Following this representation of the Radon-Nikodym derivative, we obtain that

$$\hat{\theta}_T - \theta_0 = \frac{\int_0^T J_2(t)dM_t^H}{\int_0^T J_2^2(t)dw_t^H}.$$

# Linear SDE driven by fBm

- Note that the quadratic variation $< Z >$ of the process $Z$ is the same as the quadratic variation $< M^H >$ of the martingale $M^H$ which in turn is equal to $w^H$. Hence

$$[w_T^H]^{-1} \lim_n \Sigma[Z_{t_{i+1}^{(n)}} - Z_{t_i^{(n)}}]^2 = 1 \;\text{ a.s }\; [P_{\theta_0}]$$

where $(t_i^{(n)})$ is a partition of the interval $[0, T]$ such that $\sup |t_{i+1}^{(n)} - t_i^{(n)}|$ tends to zero as $n \to \infty$.

# Linear SDE driven by fBm

- If the function $\sigma(t)$ is an unknown constant $\sigma$, the above property can be used to obtain a strongly consistent estimator of $\sigma^2$ based on the continuous observation of the process $X$ over the interval $[0, T]$. Here after we assume that the non-random function $\sigma(t)$ is known.

# Linear SDE driven by fBm

- **Theorem :** The maximum likelihood estimator $\hat{\theta}_T$ is strongly consistent, that is,

$$\hat{\theta}_T \to \theta_0 \text{ a.s } [P_{\theta_0}] \text{ as } T \to \infty$$

provided

$$\int_0^T J_2^2(t)dw_t^H \to \infty \text{ a.s } [P_{\theta_0}] \text{ as } T \to \infty.$$

# Linear SDE driven by fBm

- **Limiting distribution :** We now discuss the limiting distribution of the MLE $\hat{\theta}_T$ as $T \to \infty$.
  Let
  $$R_T = \int_0^T J_2(t) dM_t^H, T \geq 0.$$

# Linear SDE driven by fBm

- **Theorem :** Assume that the functions $b(t,s)$ and $\sigma(t)$ are such that the process $\{R_t, t \geq 0\}$ is a local continuous martingale and that there exists a norming function $I_t, t \geq 0$ such that

$$I_T^2 < R_T >= I_T^2 \int_0^T J_2^2(t) dw_t^H \xrightarrow{p} \eta^2 \text{ as } T \to \infty$$

where $I_T \to 0$ as $T \to \infty$ and $\eta$ is a random variable such that $P(\eta > 0) = 1$.

# Linear SDE driven by fBm

- Then

$$(I_T R_T, I_T^2 < R_T >) \to (\eta Z, \eta^2) \text{ in distribution as } T \to \infty$$

where the random variable $Z$ has the standard normal distribution and the random variables $Z$ and $\eta$ are independent.

# Linear SDE driven by fBm

- **Theorem :** Suppose the conditions stated earlier hold. Then

$$I_T^{-1}(\hat{\theta}_T - \theta_0) \to \frac{Z}{\eta} \text{ in distribution as } T \to \infty$$

where the random variable $Z$ has the standard normal distribution and the random variables $Z$ and $\eta$ are independent.

# Linear SDE driven by fBm

- **Remarks :** If the random variable $\eta$ is a constant with probability one, then the limiting distribution of the maximum likelihood estimator is normal with mean 0 and variance $\eta^{-2}$. Otherwise it is a mixture of the normal distributions with mean zero and variance $\eta^{-2}$ with the mixing distribution as that of $\eta$. The rate of convergence of the distribution of the maximum likelihood estimator can also be obtained.

# Linear SDE driven by fBm

- **Bayes estimation :** Suppose that the parameter space $\Theta$ is open and $\Lambda$ is a prior probability measure on the parameter space $\Theta$. Further suppose that $\Lambda$ has the density $\lambda(.)$ with respect to the Lebesgue measure and the density function is continuous and positive in an open neighbourhood of $\theta_0$, the true parameter. Let

$$\alpha_T \equiv I_T R_T = I_T \int_0^T J_2(t) dM_t^H$$

and

$$\beta_T \equiv I_T^2 < R_T >= I_T^2 \int_0^T J_2^2(t) dw_t^H.$$

# Linear SDE driven by fBm

- The maximum likelihood estimator satisfies the relation

$$\alpha_T = (\hat{\theta}_T - \theta_0) I_T^{-1} \beta_T.$$

The posterior density of $\theta$ given the observation $X^T \equiv \{X_s, 0 \leq s \leq T\}$ is given by

$$p(\theta | X^T) = \frac{\frac{dP_\theta^T}{dP_{\theta_0}^T} \lambda(\theta)}{\int_\Theta \frac{dP_\theta^T}{dP_{\theta_0}^T} \lambda(\theta) d\theta}.$$

# Linear SDE driven by fBm

- Let us write $t = I_T^{-1}(\theta - \hat{\theta}_T)$ and define

$$p^*(t|X^T) = I_T \;\; p(\hat{\theta}_T + tI_T|X^T).$$

Then the function $p^*(t|X^T)$ is the posterior density of the transformed variable $t = I_T^{-1}(\theta - \hat{\theta}_T)$. Let

$$\nu_T(t) \equiv \frac{dP_{\hat{\theta}_T + tI_T}/dP_{\theta_0}}{dP_{\hat{\theta}_T}/dP_{\theta_0}} = \frac{dP_{\hat{\theta}_T + tI_T}}{dP_{\hat{\theta}_T}} \text{ a.s.}$$

and

# Linear SDE driven by fBm

- $$C_T = \int_{-\infty}^{\infty} \nu_T(t)\lambda(\hat{\theta}_T + tI_T)dt.$$

# Linear SDE driven by fBm

- It can be checked that

$$p^*(t|X^T) = C_T^{-1}\nu_T(t)\lambda(\hat{\theta}_T + tI_T)$$

and

$$\begin{aligned}\log \nu_T(t) &= I_T^{-1}\alpha_T[(\hat{\theta}_T + tI_T - \theta_0) - (\hat{\theta}_T - \theta_0)] \\ &\quad -\frac{1}{2}I_T^{-2}\beta_T[(\hat{\theta}_T + tI_T - \theta_0)^2 - (\hat{\theta}_T - \theta_0)^2] \\ &= t\alpha_T - \frac{1}{2}t^2\beta_T - t\beta_T I_T^{-1}(\hat{\theta}_T - \theta_0) \\ &= -\frac{1}{2}\beta_T t^2.\end{aligned}$$

# Linear SDE driven by fBm

- For convenience, we write $\beta = \eta^2$. Then

$$\beta_T \to \beta \ \text{a.s} \ [P_{\theta_0}] \ \text{as} \ T \to \infty.$$

Further suppose that $K(t)$ is a nonnegative measurable function such that, for some $0 < \epsilon < \beta$,

$$\int_{-\infty}^{\infty} K(t) \exp[-\frac{1}{2} t^2(\beta - \epsilon)] dt < \infty$$

and the maximum likelihood estimator $\hat{\theta}_T$ is strongly consistent, that is,

$$\hat{\theta}_T \to \theta_0 \ \text{a.s} \quad [P_{\theta_0}] \ \text{as} \ T \to \infty.$$

# Linear SDE driven by fBm

- In addition, suppose that the following condition holds for every $\epsilon > 0$ and $\delta > 0$ :

$$\exp[-\epsilon I_T^{-2}] \int_{|u|>\delta} K(uI_T^{-1})\lambda(\hat{\theta}_T + u) du \to 0 \ \text{a.s.}[P_{\theta_0}] \ \text{as} \ T \to \infty.$$

Then we have the following theorem which is an analogue of the Bernstein - von Mises theorem.

# Linear SDE driven by fBm

- **Theorem :** Suppose that $\lambda(.)$ is a prior density which is continuous and positive in an open neighbourhood of $\theta_0$, the true parameter. Under some additional regularity conditions,

$$\lim_{T\to\infty}\int_{-\infty}^{\infty} K(t)|p^*(t|X^T)-(\frac{\beta}{2\pi})^{1/2}\exp(-\frac{1}{2}\beta t^2)|dt = 0 \ \text{ a.s } \ [P_{\theta_0}].$$

# Linear SDE driven by fBm

- As a consequence of the above theorem, we obtain the following result by choosing $K(t) = |t|^m$, for some integer $m \geq 0$.

  **Theorem :** Assume that the following conditions hold:

$$(C1)\ \ \hat{\theta}_T \to \theta_0 \ \text{ a.s } [P_{\theta_0}] \ \text{ as } \ T \to \infty,$$

$$(C2)\ \ \beta_T \to \beta > 0 \ \text{ a.s } [P_{\theta_0}] \ \text{ as } \ T \to \infty.$$

# Linear SDE driven by fBm

- Further suppose that (C3)$\lambda(.)$ is a prior probability density on $\Theta$ which is continuous and positive in an open neighbourhood of $\theta_0$, the true parameter and

$$(C4) \quad \int_{-\infty}^{\infty} |\theta|^m \lambda(\theta) d\theta < \infty$$

for some integer $m \geq 0$. Then

$$\lim_{T\to\infty} \int_{-\infty}^{\infty} |t|^m |p^*(t|X^T) - (\frac{\beta}{2\pi})^{1/2} \exp(-\frac{1}{2}\beta t^2)| dt = 0 \ \text{ a.s } \ [P_{\theta_0}].$$

# Linear SDE driven by fBm

- In particular, choosing $m = 0$, we obtain that

$$\lim_{T\to\infty} \int_{-\infty}^{\infty} |p^*(t|X^T) - (\frac{\beta}{2\pi})^{1/2} \exp(-\frac{1}{2}\beta t^2)|dt = 0 \quad \text{a.s} \quad [P_{\theta_0}]$$

  whenever the conditions (C1), (C2) and (C3) hold. This is the analogue of the Bernstein-von Mises theorem for a class of diffusion processes proved in Prakasa Rao (1981) and it shows the asymptotic convergence in the $L_1$-mean of the posterior density to the normal distribution.

# Linear SDE driven by fBm

- As a corollary , we also obtain that the conditional expectation, under $P_{\theta_0}$, of $[I_T^{-1}(\hat{\theta}_T - \theta)]^m$ converges to the corresponding $m$-th absolute moment of the normal distribution with mean zero and variance $\beta^{-1}$. We define a *regular Bayes estimator* of $\theta$, corresponding to a prior probability density $\lambda(\theta)$ and the loss function $L(\theta, \phi)$, based on the observation $X^T$, as an estimator which minimizes the posterior risk

$$B_T(\phi) \equiv \int_{-\infty}^{\infty} L(\theta, \phi) p(\theta | X^T) d\theta.$$

over all the estimators $\phi$ of $\theta$.

# Linear SDE driven by fBm

- Suppose there exists a measurable regular Bayes estimator $\tilde{\theta}_T$ for the parameter $\theta$ (cf. Theorem 3.1.3, Prakasa Rao (1987).) Suppose that the loss function $L(\theta,\phi)$ satisfies the following conditions:

$$L(\theta,\phi)=\ell(|\theta-\phi|)\geq 0$$

and the function $\ell(t)$ is nondecreasing for $t\geq 0$. An example of such a loss function is $L(\theta,\phi)=|\theta-\phi|$.

# Linear SDE driven by fBm

- Suppose there exist nonnegative functions $R(t)$, $J(t)$ and $G(t)$ such that $(D1)\ \ R(t)\ell(tI_T) \leq G(t)$ for all $T \geq 0$, and $(D2)\ \ R(t)\ell(tI_T) \to J(t)$ as $T \to \infty$ uniformly on bounded intervals of $t$. Further suppose that the function

$$(D3) \quad \int_{-\infty}^{\infty} J(t+h)\exp[-\frac{1}{2}\beta t^2]dt$$

has a strict minimum at $h = 0$. Under some additional conditions on the function $G(t)$, the following result gives the asymptotic properties of the Bayes risk of the estimator $\tilde{\theta}_T$.

# Linear SDE driven by fBm

- **Theorem :** Suppose the conditions (C1) to (C3) and the conditions (D1) to (D3) stated above hold. Then

$$I_T^{-1}(\tilde{\theta}_T - \hat{\theta}_T) \to 0 \text{ a.s } [P_{\theta_0}] \text{ as } T \to \infty$$

and

$$\begin{aligned}\lim_{T\to\infty} R(T)B_T(\tilde{\theta}_T) &= \lim_{T\to\infty} R(T)B_T(\hat{\theta}_T) \\ &= \left(\frac{\beta}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} K(t)\exp[-\frac{1}{2}\beta t^2]dt \text{ a.s } [P_{\theta_0}]\end{aligned}$$

# Linear SDE driven by fBm

- We have observed earlier that

$$I_T^{-1}(\hat{\theta}_T - \theta_0) \to N(0, \beta^{-1}) \text{ in distribution as } T \to \infty.$$

As a consequence, we obtain that

$$\tilde{\theta}_T \to \theta_0 \text{ a.s } [P_{\theta_0}] \text{ as } T \to \infty$$

and

$$I_T^{-1}(\tilde{\theta}_T - \theta_0) \to N(0, \beta^{-1}) \text{ in distribution as } T \to \infty.$$

# Linear SDE driven by fBm

- In other words, the Bayes estimator is asymptotically normal and has asymptotically the same distribution as the maximum likelihood estimator.

# Nonparametric inference

- Consider the stochastic differential equation

$$dX_t = S(X_t)\,dt + \epsilon\, dW_t^H, X_0 = x_0, 0 \leq t \leq T$$

where the function $S(.)$ is unknown. We have to estimate the function $S(.)$ based on the observation $\{X_t, 0 \leq t \leq T\}$. Suppose $\{x_t, 0 \leq t \leq T\}$ is the solution of the differential equation

$$\frac{dx_t}{dt} = S(x_t), x_0, 0 \leq t \leq T.$$

# Nonparametric inference

- Further suppose that the trend coefficient $S(x)$ satisfies the following conditions which ensure the existence and uniqueness of the solution of the SDE.
  $(A_1)$ : There exists $L > 0$ such that
  $|S(x) - S(y)| \leq L|x - y|$ whenever $|x|, |y| \in R$.
  $(A_2)$ : There exists a constant $L > 0$ such that
  $|S(x)| \leq M(1 + |x|), x \in R$.

# Nonparametric inference

- **Lemma :** Let the function $S(.)$ satisfy the conditions $(A_1)$ and $(A_2)$ and suppose that $L_N = L$ for all $N \geq 1$ . Then, with probability one,

$$(a)|X_t - x_t| < e^{Lt}\epsilon|W_t^H|$$

and

$$(b) \sup_{0 \leq t \leq T} E(X_t - x_t)^2 \leq e^{2LT}\epsilon^2 T^{2H}.$$

# Nonparametric inference

- **Proof of (a) :** Let $u_t = |X_t - x_t|$. Then, by $(A_1)$, we have

$$\begin{aligned} u_t &\leq \int_0^t |S(X_v) - S(x_v)|\, dv + \epsilon\, |W_t^H| \\ &\leq L \int_0^t u_v dv + \epsilon\, |W_t^H|. \end{aligned}$$

Applying the Gronwall's lemma, it follows that

$$u_t \leq \epsilon |W_t^H| e^{Lt}.$$

**Proof of (b) :** Check that

$$E(X_t - x_t)^2 \leq e^{2Lt}\epsilon^2 E(|W_t|^H)^2 = e^{2Lt}\epsilon^2 t^{2H}.$$

# Nonparametric inference

- Let $\Theta_0(L)$ denote the class of all functions $S(x)$ satisfying the conditions $(A_1)$ and $(A_2)$. Let $\Theta_k(L)$ denote the class of all functions $S(x)$ defined on the interval $[0, T]$ which are uniformly bounded by the same constant $C$ and which are $k$-times differentiable with respect to $x$ satisfying the condition

$$|S^{(k)}(x) - S^{(k)}(y)| \leq L|x - y|, x, y \in R.$$

Here $g^{(k)}(x)$ denotes the $k$-th derivative of $g(.)$ at x.

# Nonparametric inference

- Let $G(u)$ be a bounded function with finite support $[A, B]$ satisfying the condition $(A_3) G(u) = 0$ for $u < A$ and $u > B$, and $\int_A^B G(u)du = 1$. Suppose that the following conditions are satisfied by the function $G(.)$ :

  (i) $\int_{-\infty}^{\infty} G^2(u)du < \infty$;

  (ii) $\int_{-\infty}^{\infty} u^{2(k+1)} G^2(u)du < \infty$ , and

  (iii) $\int_{-\infty}^{\infty} (G(u))^{\frac{1}{H}} du < \infty$.

  Define a kernel type estimator of the trend $S(X_t)$ as

$$\widehat{S}_t = \frac{1}{\varphi_\epsilon} \int_0^T G\left(\frac{\tau - t}{\varphi_\epsilon}\right) dX_\tau$$

# Nonparametric inference

- **Theorem :** Suppose that the trend function $S(x) \in \Theta_0(L)$ and the function $\varphi_\epsilon \to 0$ such that $\epsilon^2 \varphi_\epsilon^{-1} \longrightarrow 0$ as $\epsilon \to 0$. Suppose the conditions $(A_1), (A_2)$ and $(A_3)$ are satisfied. Then, for any $0 < c \leq d < T$, the estimator $\widehat{S}_t$ is uniformly consistent, that is ,

$$\lim_{\epsilon \to 0} \sup_{S(x) \in \Theta_0(L)} \sup_{c \leq t \leq d} E_S(|\widehat{S}_t - S(x_t)|^2) = 0.$$

# Nonparametric inference

- In addition to the conditions $(A_1)-(A_3)$, assume that $(A_4) \int_{-\infty}^{\infty} u^j G(u)du = 0$ for $j = 1, 2, ...k$; $\int_{-\infty}^{\infty} |G(U)u^{k+1}|du < \infty$.
  **Theorem :** Suppose that the function $S(x) \in \Theta_{k+1}(L)$ and $\varphi_\epsilon = \epsilon^{\frac{1}{k-H+2}}$. Then, under the conditions $(A_1), (A_2), (A_3)$ and $(A_4)$,

$$\limsup_{\epsilon \to 0} \sup_{S(x) \in \Theta_{k+1}(L)} \sup_{c \le t \le d} E_S(|\widehat{S}_t - S(x_t)|^2)\epsilon^{\frac{-2(k+1)}{k-H+2}} < \infty.$$

# Nonparametric inference

- **Theorem :** Suppose that the function $S(x) \in \Theta_{k+1}(L)$ and $\varphi_\epsilon = \epsilon^{\frac{1}{k-H+2}}$. Then, under the conditions $(A_1), (A_2), (A_3)$ and $(A_4)$, the asymptotic distribution of

$$\epsilon^{\frac{-(k+1)}{k-H+2}}(\widehat{S}_t - S(x_t))$$

is Gaussian with the mean

$$m = \frac{S^{(k+1)}(x_t)}{(k+1)!} \int_{-\infty}^{\infty} G(u) u^{k+1} \, du$$

and the variance

$$\sigma^2 = H(2H-1) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(u) G(v) |u-v|^{2H-2} du \, dv$$

# References


- (1) B.L.S. Prakasa Rao, *Asymptotic Theory of Statistical Inference*, John Wiley, New York (1987).
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